Combinatorics in Banach space theory (MIM UW 2014/15)

PROBLEMS (Part 2)

PROBLEM 2.1. Give a (short) proof of the following statement, called *Rosenthal's lemma* (1970): Let Σ be a σ-algebra of subsets of Ω and $(\mu_n)_{n=1}^{\infty}$ a uniformly bounded sequence of finitely additive, non-negative measures on Σ. Then, for every pairwise disjoint sequence $(E_n)_{n=1}^{\infty} \subset \Sigma$ and every $\varepsilon > 0$ there exists a strictly increasing sequence of indices $(n_k)_{k=1}^{\infty}$ such that

$$\mu_{n_k}\Bigg(\bigcup_{j\neq k} E_{n_j}\Bigg) < \varepsilon \quad \text{for each } k \in \mathbb{N}.$$

Hint. Take any partition $\bigcup_{p=1}^{\infty} M_p$ of $\mathbb{N}$ consisting of pairwise disjoint infinite subsets of $\mathbb{N}$ and consider two cases; (a): when there is $p \in \mathbb{N}$ for which $\mu_k(\bigcup_{j\in M_p,\, j\neq k} E_j) < \varepsilon$ for every $k \in M_p$, and (b): otherwise.

PROBLEM 2.2. Use Rosenthal's lemma in order to give a (relatively short) proof of *Phillips' lemma* (1940): Let $(\mu_n)_{n=1}^{\infty}$ be a uniformly bounded sequence of finitely additive, scalar-valued measures on $2^{\mathbb{N}}$. If for every set $E \subset \mathbb{N}$ we have $\lim_{n\to\infty} \mu_n(E) = 0$, then

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} |\mu_n(\{k\})| = 0.$$

Remark. In fact it is enough to assume that each μ_n is bounded, since then the uniform boundedness follows from the so-called *Nikodým boundedness principle.*

PROBLEM 2.3. By using Phillips' lemma, show that ℓ_1 has the Schur property (that is, weakly convergent sequences converge in norm).

PROBLEM 2.4. Verify that Phillips' lemma may be equivalently stated by saying that the canonical projection $\pi\colon c_0^{***} \to c_0^*$ is sequentially weak*-to-norm continuous. By the canonical projection (called also the *Dixmier projection*) from X^{***} onto X^* we mean the one given by $\pi(x^{***}) = x^{***}|_{j(X)}$, where $j\colon X \to X^{**}$ is the canonical embedding. We say that a Banach space X has the [*weak*] *Phillips property* whenever the Dixmier projection from X^{***} onto X^* is sequentially weak*-to-norm [weak*-to-weak] continuous. Show that for every Banach space X with the weak Phillips property the dual X^* is weakly sequentially complete (that is, every weakly Cauchy sequence is weakly convergent).

Remark. In view of this assertion, we may say that Phillips' lemma is responsible for ℓ_1 being weakly sequentially complete.

PROBLEM 2.5. Let X be a Banach space. Show how Rosenthal's ℓ_1-theorem implies that each of the following two assumptions:

(a) X is weakly sequentially complete and non-reflexive,
(b) X is infinite-dimensional with the Schur property,

forces X to contain an isomorphic copy of ℓ_1.

PROBLEM 2.6. Explain why the (part of the) Odell–Rosenthal theorem may be as well stated in the following way: A separable Banach space X does not contain an isomorphic copy of ℓ_1 if and only if every bounded set $A \subset X$ is sequentially weak* dense in its weak* closure (in X^{**}). (The difference between this formulation and the one discussed during lectures is that we allow A to be any bounded set, not necessarily B_X.) Decide whether such a statement holds true if we drop the word '*bounded*'.

PROBLEM 2.7. By the Bourgain–Fremlin–Talagrand theorem (1978), for every separable Banach space X the condition that $\ell_1 \hookrightarrow X$ (and hence any on the list by Odell and Rosenthal) is equivalent to $(B_{X^{**}}, w^*)$ containing a homeomorphic copy of $\beta\mathbb{N}$. Verify this statement, by giving concrete constructions of copies of $\beta\mathbb{N}$, in the following two cases:

(a) $X = \ell_1$,
(b) $X = C[0,1]$ (here, $\ell_1 \hookrightarrow X$ follows from the Banach–Mazur theorem).

Remark. The proof that $\ell_1 \hookrightarrow X$ implies $\beta\mathbb{N} \hookrightarrow (B_{X^{**}}, w^*)$ homeomorphically requires the following deep statement, called *Rosenthal's dichotomy*: given a pointwise bounded sequence $(f_n)_{n=1}^{\infty}$ of continuous functions on a Polish space D, either it contains a pointwise convergent subsequence, or a subsequence whose closure in $\mathbb{R}^D$ is homeomorphic to $\beta\mathbb{N}$ (consult, e.g., Chapter 1 in [S. Todorcevic, *Topics in Topology*, Springer 1997]). The proof of the converse implication is very easy: $\mathrm{card}(\beta\mathbb{N}) = 2^{\mathfrak{c}}$, while Odell and Rosenthal have already taught us that the latter condition is equivalent to $\ell_1 \hookrightarrow X$.

PROBLEM 2.8. Let X be a separable Banach space. Prove that $\ell_1 \hookrightarrow X$ if and only if $C[0,1]$ is isomorphic to a quotient of X. Next, give an example showing that this is no longer true for non-separable spaces.

Hint. You can use the fact that ℓ_1 is a projective object in the category of Banach spaces, that is, it enjoys the following *lifting property* (quite easy to prove): For all Banach spaces X and Y for which there is a surjective operator $T\colon X \to Y$, and for every operator $U\colon \ell_1 \to Y$ there exists a *lifting* of U, i.e. an operator $S\colon \ell_1 \to X$ satisfying $TS = U$.

In the proof of the '*if*' part you may use the following Pełczyński theorem: If a separable Banach space X contains a subspace Y isomorphic to $C[0,1]$, then Y contains a further subspace Z that is still isomorphic to $C[0,1]$ and also complemented in X.

Remark. This was proved by Pełczyński in 1968 and gives a very efficient way of producing some badly behaved sets in the dual of a Banach space containing ℓ_1. You shall see it in the next few exercises which provide some interesting consequences of the Odell–Rosenthal theorem.

PROBLEM 2.9. Let X be a Banach space. Recall that given bounded sets $B \subseteq C \subset X^*$ we call B a *James boundary of* C provided that for every $x \in X$ there exists $f_0 \in B$ so that

$$f_0(x) = \sup\{f(x)\colon f \in C\}.$$

Moreover, $B \subset B_{X^*}$ is called a *James boundary of* X if it is a James boundary of B_{X^*}. Prove the following statements:

(a) The set $\mathrm{ext}(C)$ of extreme points of C forms a James boundary of C, whenever $C \subset X^*$ is w^*-compact;
(b) There is a James boundary B (for example, in the space $\ell_1(\Gamma)$ with Γ uncountable) such that $B \cap \mathrm{ext}(B_{X^*}) = \varnothing$;
(c) On the other hand, if B is a James boundary of any Banach space X, then $\mathrm{ext}(B_{X^*}) \subseteq \overline{B}^{w^*}$.

PROBLEM 2.10. Let X be a separable Banach space. Prove that the following assertions are equivalent:

(i) $\ell_1 \not\hookrightarrow X$;
(ii) for every closed, convex and bounded set $C \subset X^*$ we have the equivalence: C is w^*-compact iff every $x \in X$ (regarded as a functional on X^*) attains its supremum on C.

Hint. (1) Of course, in view of the Banach–Alaoglu theorem, the proof of '(i) $\Rightarrow$ (ii)' is all about the weak* closedness of C. Use the Hahn–Banach separation theorem and Simon's inequality which reads as follows: If $B \subset X^*$ is a boundary of some bounded set in X^* and $(x_n)_{n=1}^\infty \subset X$ is any bounded sequence, then

$$\sup_{y^*\in B}\Big(\limsup_{n\to\infty}\langle x_n, y^*\rangle\Big) \geqslant \inf\Big\{\sup_B x\colon\, x \in \operatorname{conv}\{x_n\}_{n=1}^\infty\Big\}.$$

Observe that the assumption of the '*if*' part in the desired equivalence say nothing but C is its own James boundary. **(2)** Apply Problem 2.8 and think about the set of all atomic probabilistic measures on $[0, 1]$.

Remark. The above statement characterizes Banach spaces for which the "*weak**-*James theorem*" holds true.

PROBLEM 2.11. By giving a concrete example, show that the assertion of Problem 2.10 is not true if we do not assume that X is separable.

PROBLEM 2.12. Let X be a separable Banach space. Prove that the following assertions are equivalent:

(i) $\ell_1 \not\hookrightarrow X$;
(ii) for every weak* compact and convex set $K \subset X^*$ we have $K = \overline{\operatorname{conv}}^{\|\cdot\|}(\operatorname{ext}(K))$.

Hint. (1) Use the Choquet representation theorem: If D is a metrizable, compact, convex subset of a locally convex linear topological space $\mathcal{E}$, then every point $x_0 \in D$ is *representable* by some probability measure μ on D with support in $\operatorname{ext}(D)$, that is, for every $\varphi \in \mathcal{E}^*$ we have $\varphi(x_0) = \int_D \varphi\,\mathrm{d}\mu$. Recall also that (i) implies some nice properties for all functionals from X^{**}. **(2)** Again, make use of Problem 2.8.

Remark. Note that if we replace the norm closure by the weak* closure in condition (ii) we get nothing else but the Krein–Milman theorem which is true whether or not X contains ℓ_1.

PROBLEM 2.13. Let X be a Banach space not containing an isomorphic copy of ℓ_1. Show that for every equivalent norm $\|\cdot\|$ on X the intersection of two 1-norming hyperplanes of X^* is again 1-norming. (Recall that $Z \subset X^*$ is called 1-*norming* whenever $\|x\| = \sup_{z^*\in Z} z^*(x)$ for each $x \in X$.)

Hint. Use the part of the Odell–Rosenthal theorem which says that in our situation every functional from $B_{X^{**}}$ is a Baire 1 function on the Polish space (B_{X^*}, w^*), and combine it with the Baire theorem saying that every Baire 1 function (defined on a metric separable space and with values in a separable space) is continuous on some dense G_δ-subset of its domain.

Remark. The property of 1-norming hyperplanes stated above is also a sufficient condition for X not containing ℓ_1. One can also replace the word '*hyperplanes*' by '*subspaces*'. This was proved by Godefroy and Kalton (1989).

PROBLEM 2.14. Let X and Y be Banach spaces such that X^* and Y^* are isometrically isomorphic and contain no isomorphic copy of ℓ_1. Show that X and Y are isometrically isomorphic.

Hint. X and Y may be viewed as subspaces of the same space Z^*, so that both X and Y are preduals of Z. We then have $Z^{**} = Z \oplus X^{\perp} = Z \oplus Y^{\perp}$ (why?). Use also the assertion of Problem 2.13.

Remark. Of course, after dropping the assumption that X^* and Y^* do not contain ℓ_1 the assertion fails drastically—we know, for example, that ℓ_1 itself has uncountably many pairwise non-isomorphic preduals.